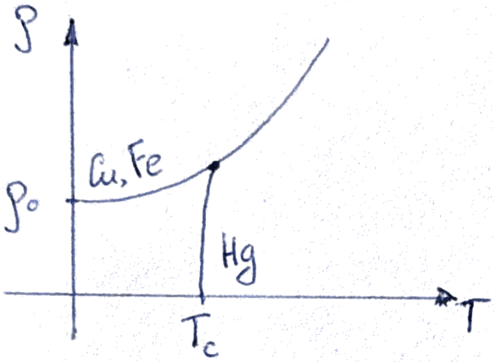
1. Introduction to superconductivity

1.1. Discovery of superconductivity and first experiments

In 1908, Kamerlingh-Onnes obtained liquid ^4He ($T_{boil} = 4.2\,K$)

In 1911, he discovered superconductivity (SC) in Hg.

At low temperatures, electrical resistivity: $\rho \simeq \underbrace{\rho_0}_{\text{defects, zero-oscillation}} + \underbrace{B \cdot T^5}_{\text{contribution from phonons}}$



For very pure Cu @ $T = 4.2\,K$

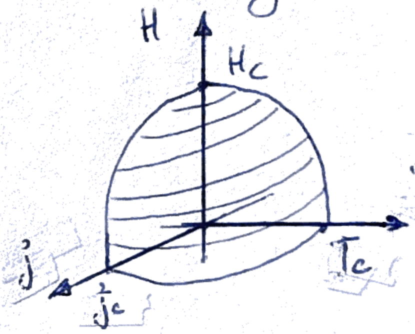
$\rho \simeq 10^{-9}\ Ohm \cdot cm$

For a SC: $\rho \lesssim 10^{-24}\ Ohm \cdot cm$

In 1914, he discovered that SC is destroyed in a magnetic field $B > B_c$ (B_c or H_c - critical field).

Strong current, $j > j_c$, also destroys SC.

Phase diagramm



Note: $\underbrace{\vec{B}}_{\substack{\text{total field} \\ [T]}} = \underbrace{\mu_0}_{\substack{\text{vacuum magnet.} \\ \text{permeability}}} \cdot (\underbrace{\vec{H}}_{\substack{\text{external field} \\ [A/m]}} + \underbrace{\vec{M}}_{\text{magnetization}})$

$\mu_0 \simeq 4\pi \cdot 10^{-7}\ N/m$

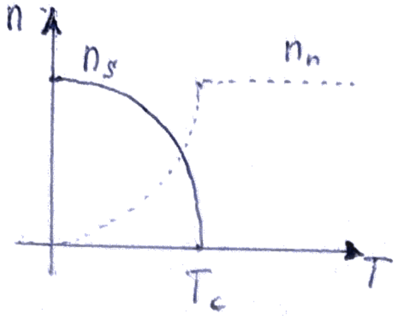
	T_c [K]	B_c [T]		j_c [A/cm^2]
Hg	4	0.04	I-type	
Sn	3.7	0.03		$\sim 10^4$
Pb	7.2	0.08		
Nb	9.3	0.21	II-type (B_{c2})	
Nb_3Sn (1954)	18	27		$\sim 10^6$
MgB_2	39	39		

	T_c [K]	B_c [T]		j_c [A/cm^2]
(1986) $La_{2-x}Ba_xCuO_4$, $x \sim 0.15-0.2$	30-35		II-type (B_{c2})	
(1987) $YBa_2Cu_3O_{7-x}$	~92	92		
(2026) $HgBa_2Ca_2Cu_3O_{8+x}$	151			$\sim 10^7$
(2019) LaH_{10} ($P = 151\,GPa$)	250	~100		

for Cu wire $j_{max} = \frac{10A}{1.31\,mm^2} = 760 \frac{A}{cm^2}$. Cu melts at $j_{max} \sim 10^4 \frac{A}{cm^2}$

1.2. Phenomenological model of Casimir and Gorter (1930-1933)

There are two types of electrons: „normal" with concentration n_n
„superconducting" n_s



„superconducting" electrons do not scatter and transfer current without dessipation

$$m \cdot \frac{d\vec{V}_s}{dt} = e\vec{E}, \quad \vec{j}_s = e \cdot n_s \cdot \vec{V}_s$$

$$\frac{d\vec{j}_s}{dt} = \frac{e^2 n_s \vec{E}}{m}$$

Drude model for normal electrons

$$m\frac{d\vec{V}_n}{dt} = e\vec{E} - \frac{m\vec{V}_n}{\tau} \Rightarrow \vec{j}_n = \frac{e^2 n_n \tau \vec{E}}{m}$$

Ampere-Maxwell law: $\nabla \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t} \xrightarrow[\text{magnet. dipoles}]{\text{no elect.}} \nabla \times \vec{B} = \mu_0 \cdot \vec{j}$

$$\frac{\partial}{\partial t} \nabla \times \vec{B} = \mu_0 \cdot \frac{\partial \vec{j}_s}{\partial t} = \frac{e^2 \mu_0 n_s \vec{E}}{m}$$

Maxwell-Faraday law: $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

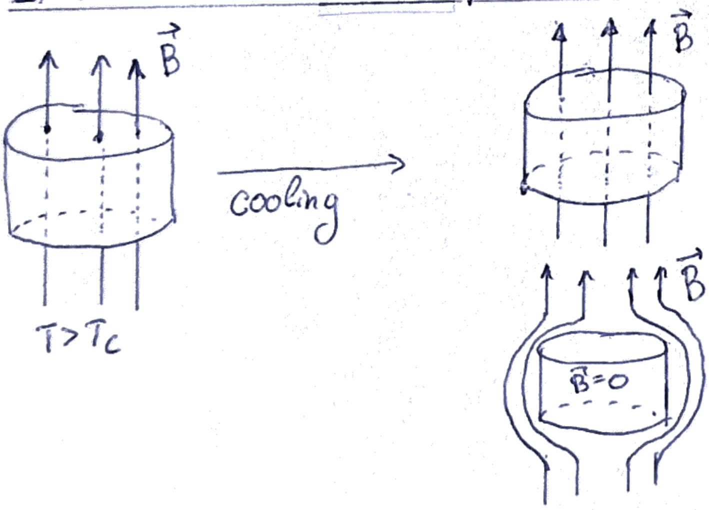
$$\Rightarrow \frac{\partial}{\partial t} \nabla \times \nabla \times \vec{B} = \frac{e^2 \mu_0 n_s}{m} \cdot \left(-\frac{\partial \vec{B}}{\partial t}\right)$$

$$\frac{\partial}{\partial t}\left[\nabla \times \nabla \times \vec{B} + \frac{e^2 \mu_0 n_s}{m} \vec{B}\right] = 0$$

$\Rightarrow \vec{B} = const$ (does not depend on time)

Casimir and Gorter model describes an <u>ideal conductor</u>, for which $\vec{B} = const$.

1.3. Meissner-Ochsenfeld effect (1933)

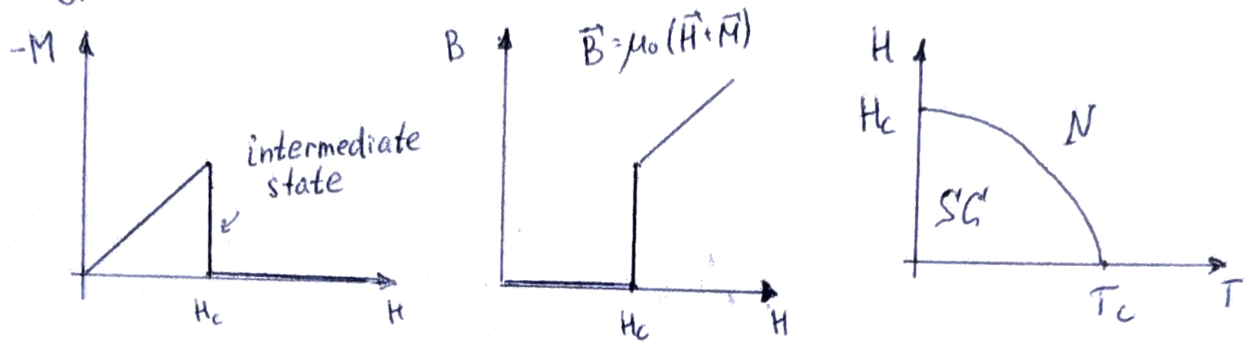


- ideal conductor (was not observed)

- superconductor, $\boxed{\vec{B}=0}$ ← meissner phase (ideal diamagnetic)

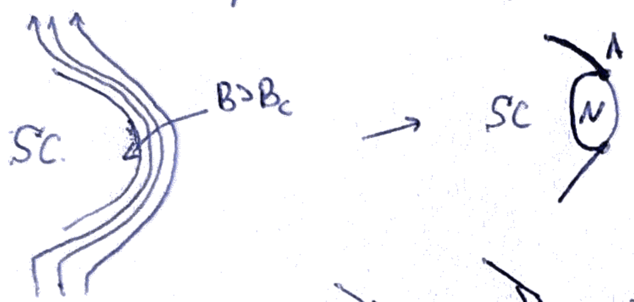
1.4 Two types of superconductors

Type-I:

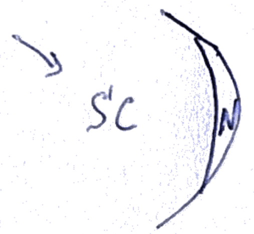


Intermediate state: alternating SC and N domains.

It is not possible to form a single simply-connected N-domain in a SC.

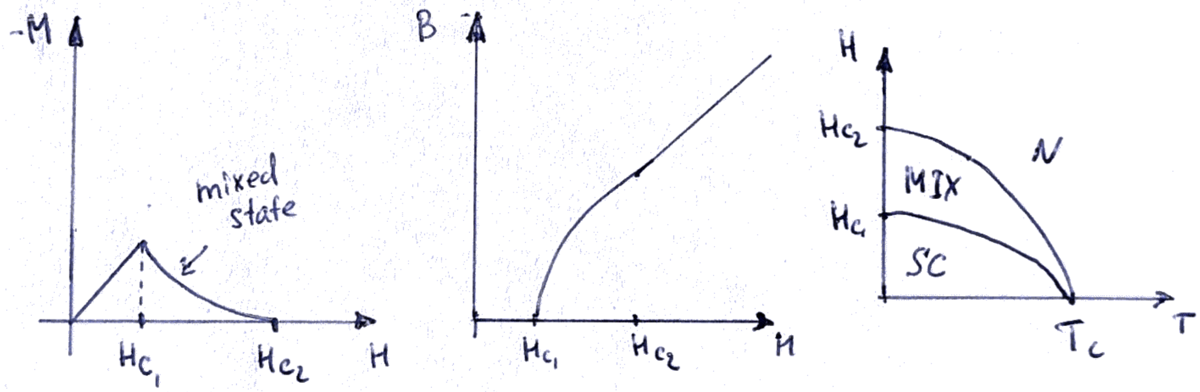


field lines are parallel to the border between SC and N. Field lines would need to break at the point A, which is impossible

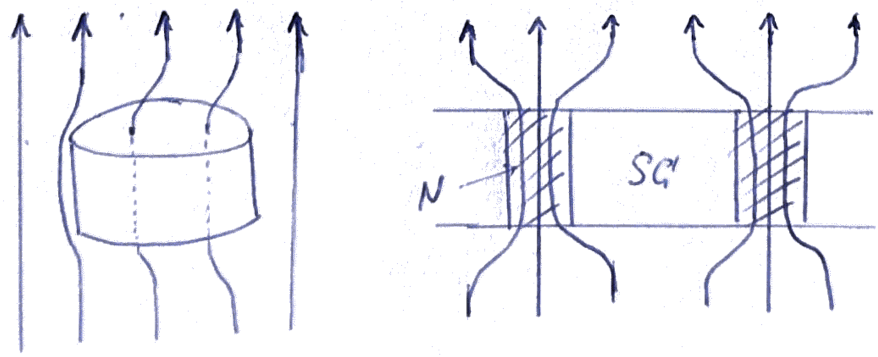


in this case $B<B_c$ in the N-phase, so it should turn into the SC-phase.

Type-II:



Mixed state: single magnetic vortices go into the SC phase.
The supercurrent circles arond vortices (Shubnikov's phase)



1.5. Phenomenological model of Heinz London and Fritz London (1935)

From Casemir-Gorter model: $\underbrace{\nabla\times\nabla\times\vec{B}}_{\mu_0\vec{j}_s} + \frac{e^2\mu_0 n_s}{m}\vec{B} = const$

$$\nabla\times\vec{j}_s + \frac{e^2 n_s}{m}\vec{B} = const$$

Current density: $\vec{j} = q\cdot n\cdot\vec{v} = \frac{q}{2m}\cdot\vec{p}\cdot n$

Symmetric expession in quantum mechanics: $\vec{j} = \frac{q}{2m}\left[\psi^*\hat{\vec{p}}\psi + \psi(\hat{\vec{p}}\psi)^*\right]$

$\psi(\vec{r})$ - wave function of supereconducting electrons $(|\psi|^2 = n_s)$

$\hat{\vec{p}} = -i\hbar\nabla - q\vec{A}$ - canonical momentum

$\vec{A}$ - magnetic vector-potential $\begin{pmatrix}\nabla\times\vec{A} = \vec{B} \\ \vec{E} = -\nabla\varphi - \frac{\partial\vec{A}}{\partial t}\end{pmatrix}$

$$\vec{j} = \frac{q}{2m}\left[\psi^*(-i\hbar\nabla - q\vec{A})\psi + \psi(i\hbar\nabla - q\vec{A})\psi^*\right] = \underbrace{\frac{q\hbar}{2m}i(\psi\nabla\psi^* - \psi^*\nabla\psi)}_{\text{paramagnetic contribution}} - \underbrace{\frac{q^2}{m}|\psi|^2\cdot\vec{A}}_{\text{diamagnetic contribution}}$$

Londons suggested a „rigid" wavefunction, $\nabla\psi = 0$:

$$\vec{j}_s = -\frac{q^2}{m}|\psi|^2\vec{A}$$

The supercurrent is transferred by pairs of electrons (Cooper pairs):

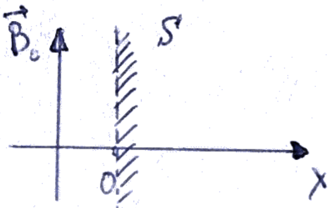
$$\begin{cases} q \to 2e \\ m \to 2m \\ |\psi|^2 \to \frac{n_s}{2} \end{cases} \Rightarrow \vec{j}_s = -\frac{e^2}{m}n_s\cdot\vec{A} \text{ - London's equation}$$

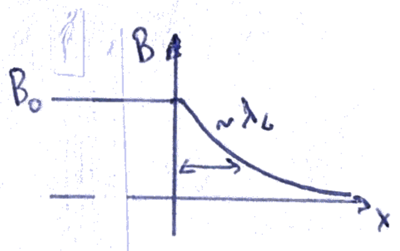
$$\begin{cases} \nabla\times\vec{j}_s + \frac{e^2 n_s}{m}\vec{B} = const \\ \vec{j}_s = -\frac{e^2 n_s}{m}\vec{A} \end{cases} \Rightarrow \quad const = 0.$$

$$\nabla\times\nabla\times\vec{B} + \frac{e^2\mu_0 n_s}{m}\vec{B} = 0$$

$$\nabla\times\nabla\times\underbrace{\nabla\times\vec{B}}_{\mu_0\vec{j}_s} + \frac{e^2\mu_0 n_s}{m}\underbrace{\nabla\times\vec{B}}_{\mu_0\vec{j}_s} = 0$$

$$\nabla\times\nabla\times\vec{j}_s + \frac{e^2\mu_0 n_s}{m}\vec{j}_s = 0$$





$$\nabla\times\nabla\times\vec{B} + \frac{e^2\mu_0 n_s}{m}\vec{B} = 0$$

$$\nabla\times\nabla\times\vec{B} = \nabla(\underbrace{\nabla\vec{B}}_{=0}) - \nabla^2\vec{B} = -\nabla^2\vec{B}$$

$$-\frac{d^2B}{dx^2} + \underbrace{\frac{e^2\mu_0 n_s}{m}}_{1/\lambda_L^2}B = 0$$

$$\frac{d^2B}{dx^2} - \frac{1}{\lambda_L^2}\cdot B = 0$$

$$B(x) = B_0 e^{-x/\lambda_L}$$

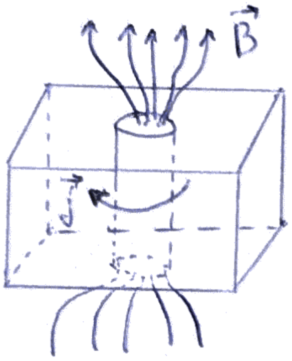
$\lambda_L = \sqrt{\frac{m}{e^2\mu_0 n_s}} \sim 10^3\,\text{Å}$ - London penetration depth

Mag. field and current only exist at the surface $(\sim\lambda_L)$ of a SC.

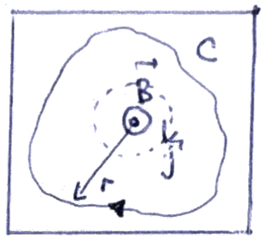
1.6. Quantization of magnetic flux (fluxoids) in a II-type SC.

Wave function of Cooper pairs $\psi(\vec{r}) = \sqrt{\frac{n_s}{2}}\cdot e^{i\theta(\vec{r})} = \psi_0 \cdot e^{i\theta(\vec{r})}$

$$\vec{j} = \frac{(2e)\hbar}{2(2m)}\, i\,(\underbrace{\psi\nabla\psi^*}_{-i|\psi|^2\nabla\theta} - \underbrace{\psi^*\nabla\psi}_{-i|\psi|^2\nabla\theta}) - \frac{(2e)^2}{(2m)}|\psi|^2\cdot\vec{A} = \frac{e\hbar}{m}|\psi_0|^2\cdot\nabla\theta - \frac{2e^2}{m}|\psi_0|^2\cdot\vec{A}$$



top view:



$r \gg \lambda_L$ (no current)

$$0 = \oint \vec{j}\,d\vec{l} = \frac{e\hbar}{m}|\psi_0|^2\cdot \underbrace{\oint \nabla\theta\cdot d\vec{l}}_{2\pi\cdot n} - \frac{2e^2}{m}|\psi_0|^2\cdot \underbrace{\oint \vec{A}\,d\vec{l}}_{\int \underbrace{(\nabla\times\vec{A})}_{=\vec{B}}\cdot d\vec{S}\ \text{(Stokes theorem)}}$$

$$2\pi n\hbar = 2e\cdot\Phi \Rightarrow \Phi = \frac{2\pi\hbar}{2e}\cdot n = \Phi_0\cdot n \qquad \Phi = \int \vec{B}\,d\vec{S} \text{ - magnetic flux.}$$

$$\Phi_0 = \frac{h}{2e} \approx 2.07\cdot 10^{-15}\left[\frac{T}{m^2}\right] = 2.07\cdot 10^{-15}[Wb]$$

Quantization of magnetic flux is equivalent to quantization of electrons in an atom.

In a normal metal, the current is transferred by single electrons. There $\Phi_0^N = \frac{h}{e}$ (see Aharonov-Bohm effect).

1.7 Phenomenological theory of Ginzburg-Landau

$\psi(\vec{r})$ → complex-value order parameter, describing the 2^{nd} order phase transition ($SC \leftrightarrow N$)

Gibb's free energy per volume (in a magnetic field)

$$g_s = g_n + \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{1}{2m}|(\hat{\vec{p}} - q\vec{A})\psi|^2 + \frac{|\vec{B}_{ext} - \vec{B}|^2}{2\mu_0}$$

$\alpha = \alpha_0\left(\frac{T}{T_c} - T\right)$ it can be shown that $\alpha = -\beta \cdot n_s$

$\beta = const > 0.$ $H_c^2 = \frac{\mu_0 \alpha^2}{\beta}$

Full energy: $G_s = \int g_s \cdot dV$. From the conditions $\frac{\delta G_s}{\delta \psi} = 0$ and $\frac{\delta G_s}{\delta \psi^*} = 0$, one can obtain:

$$\alpha\psi + \beta\psi|\psi|^2 + \frac{1}{2m}(\vec{p} - q\vec{A})^2\psi = 0 \quad (1^{st} \text{ equation})$$

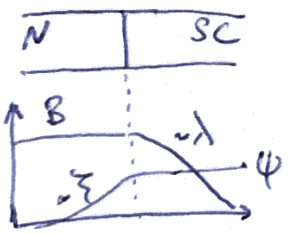
$$\vec{j} = \frac{q\hbar}{2m} i(\psi\nabla\psi^* - \psi^*\nabla\psi) - \frac{q^2}{m}|\psi|^2\vec{A} \quad (2^{nd} \text{ equation})$$

Analysis of GL equations shows, that there are two characteristic lengthscales: $\lambda = \sqrt{\frac{m\beta}{e^2\mu_0|\alpha|}} \sim \frac{1}{\sqrt{T_c - T}}$ – penetration depth

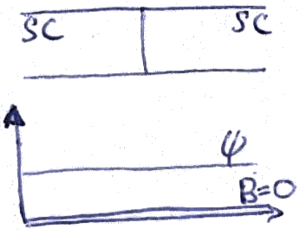
$\xi = \sqrt{\frac{\hbar^2}{4m|\alpha|}} \sim \frac{1}{\sqrt{T_c - T}}$ – coherence length (characteristic lengthscale over which ψ changes)

Surface energy of N-SC border

Competition between two states in magnetic field ($B = B_c$)



vs



surface energy:

$$\Delta G \sim \xi \cdot \alpha|\psi|^2 - \frac{B_c^2}{2\mu_0}\lambda$$

$\Delta G < 0$ – border is energetically favorable

$\Delta G > 0$ – border is energetically unfavorable

Landau-Ginzburg parameter: $\varkappa = \frac{\lambda}{\xi}$

$\varkappa = \frac{\lambda}{\xi} > \frac{1}{\sqrt{2}}$ → SC of the II-type (Shubnikov's phase with vortices)

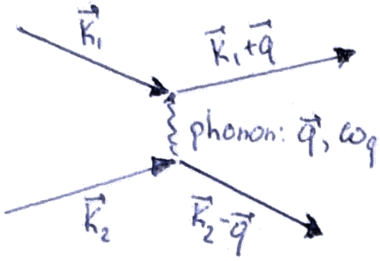
$\varkappa = \frac{\lambda}{\xi} < \frac{1}{\sqrt{2}}$ → SC of the I-type (Meissner's phase)

2. Microscopic theory of superconductivity

(Bardeen, Cooper, Schrieffer, 1957)

Isotop-effect: $T_c \sim \frac{1}{M^\alpha}$, $H_c \sim \frac{1}{M^\alpha}$, $\alpha \sim 0.5$
(1950) (for HTSC, $\alpha \sim 0.16$)

It became clear that electrons can attract to each other via lattice



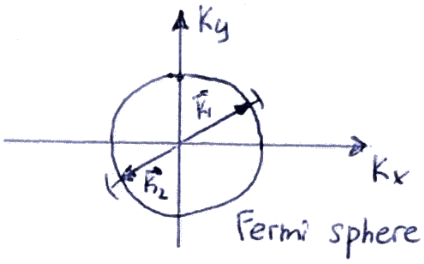
$\varepsilon_{k_1} - \varepsilon_{k_2} = \hbar\omega$

effective interaction: $V_{eff}(\vec{k}_1, \vec{k}_2) \propto \frac{4\pi e^2}{q^2+k_0^2} \cdot \frac{\omega^2}{\omega^2-\omega_q^2}$

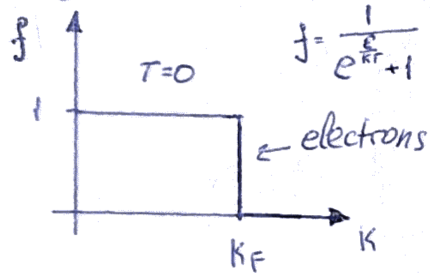
if $\omega < \omega_q \Rightarrow V_{eff} < 0$ - attraction

Weak attraction between two electrons close to the Fermi surface with $\vec{k}_1 \approx -\vec{k}_2$ and opposite spins.

Even a small attraction in 2D (Fermi surface) can lead to a bound state - a cooper pair.



The ground state of normal metal



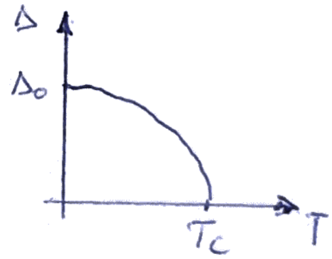
In BCS theory, the binding energy of all cooper pairs is -2Δ.

$$\Delta(T=0) = \Delta_0 = 2 \cdot \hbar\omega_D \cdot e^{-\frac{1}{N(0)\cdot V}}$$

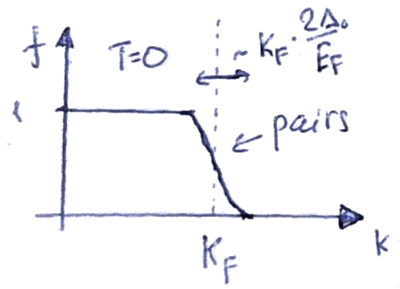
$$\Delta_0(Pb) \approx 1.36 \cdot 10^{-3}\, eV$$

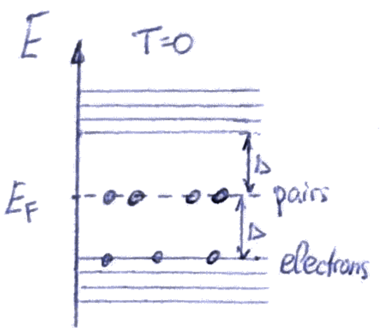
$$k_B T_c = 1.134 \cdot \hbar\omega_D \cdot e^{-\frac{1}{N(0)V}}$$

$$\Delta \approx 3.06\, k_B \sqrt{T_c(T_c-T)}$$

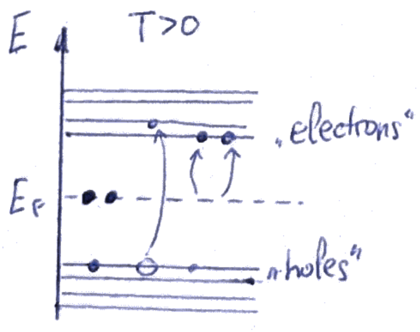


The ground state of SC:





Ground state

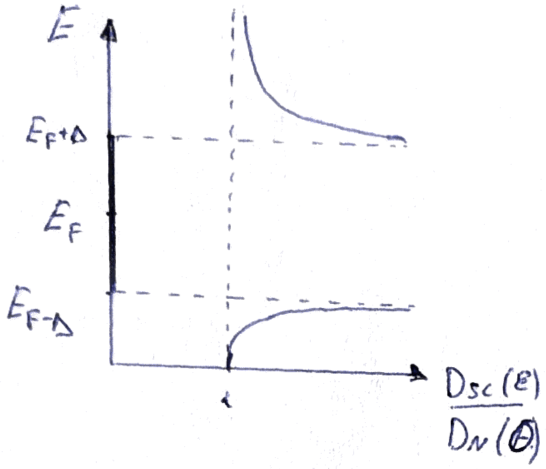


Excited state

The density of states for quasiparticles:

$\varepsilon = E - E_F$, $\quad D_{sc}(\varepsilon) = D_N(0) \cdot \frac{|\varepsilon|}{\sqrt{\varepsilon^2 - \Delta^2}}$, $\quad |\varepsilon| \geq \Delta$

$D_{sc}(\varepsilon) = 0, \quad |\varepsilon| < \Delta$



Let us estimate

a) The size of a pair:

$$\Delta = \delta\left(\frac{p^2}{2m}\right) = \frac{p_F}{m} \delta p = v_F \cdot \delta p$$

$k_F \sim 10^8\ cm^{-1}$

$\varepsilon_F \sim 10\,eV$

$\Delta \sim 0.001\,eV$

$$\Delta x \cdot \Delta p \sim \hbar \Rightarrow \Delta x \sim \frac{\hbar}{\Delta p} \sim \frac{\hbar v_F}{\Delta} \sim \frac{\varepsilon_F}{k_F \cdot \Delta} \sim \sim 10^{-4} cm \sim 10^4 \mathring{A}$$

$\hbar k = mv$

b) Concentration of pairs:

$$n_s \sim n \cdot \frac{\Delta k}{k_F} \sim n \cdot \frac{\Delta}{E_F} \sim 10^{23} cm^{-3} \cdot \frac{10^{-3} eV}{10\,eV} \sim 10^{19} cm^{-3}$$

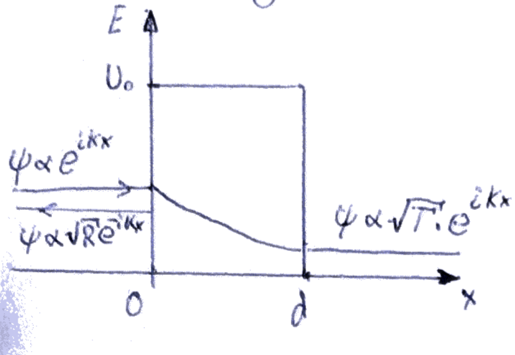
c) Number of pairs within the volume of a single pair

$$\Delta x^3 \cdot n_s \sim 10^{-12} cm^3 \cdot 10^{19} cm^{-3} \sim 10^7$$

In SC pairs are „connected" to each other. Single pair can not „relax" alone, and 'relaxation' of all pairs is energetically unfavorable. Hence, the superconducting current flows without relaxation.

3. Contact effects

3.1 Tunneling under potential barrier



Recap from quantum mechanics:

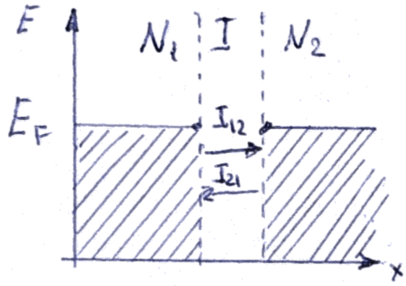
Transmission coefficient (tunneling probability)

$$T = \left(\frac{4k\varkappa}{k^2+\varkappa^2}\right)^2 \cdot e^{-2\varkappa d}, \qquad \varkappa = \sqrt{\frac{2m}{\hbar^2}(U_0-E)}$$

3.2 Contact between two metals (N-I-N)

$V=0 \Rightarrow$ no external voltage

Fermi levels of two metals will equillibrate



$\mathcal{E} = E - E_F$

Current $1 \to 2$

$$I_{12} \propto \int_{-\infty}^{+\infty} T(\mathcal{E})\, D_1(\mathcal{E})\cdot f(\mathcal{E})\cdot D_2(\mathcal{E})(1-f(\mathcal{E}))\cdot d\mathcal{E}$$

$D(\mathcal{E})$ - density of states

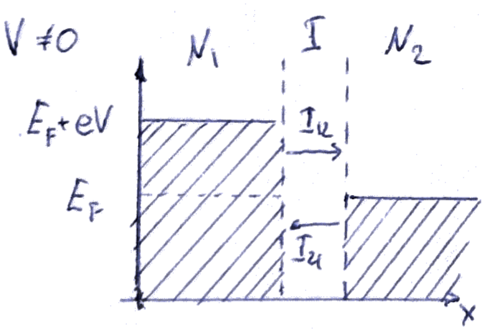
$$f(\mathcal{E}) = \frac{1}{e^{\frac{\mathcal{E}}{kT}}+1}$$ - distribution function

Current $2 \to 1$

$$I_{21} \propto \int_{-\infty}^{+\infty} T(\mathcal{E})\cdot D_2(\mathcal{E})\cdot f(\mathcal{E})\cdot D_1(\mathcal{E})(1-f(\mathcal{E}))\cdot d\mathcal{E}$$

Total current $I = I_{12} - I_{21} = 0$

$V \neq 0$



$$I_{12} \propto \int_{-\infty}^{+\infty} T(\mathcal{E})\, D_1(\mathcal{E}+eV) f(\mathcal{E}+eV)\cdot D_2(\mathcal{E})(1-f(\mathcal{E}))\cdot d\mathcal{E}$$

$$I_{21} \propto \int_{-\infty}^{+\infty} T(\mathcal{E})\, D_2(\mathcal{E})\cdot f(\mathcal{E})\cdot D_1(\mathcal{E}+eV)\cdot(1-f(\mathcal{E}+eV))\, d\mathcal{E}$$

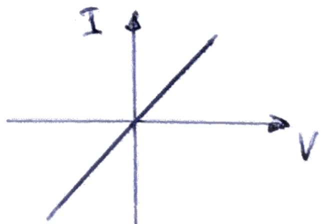
$$I = I_{12} - I_{21} \propto \int_{-\infty}^{+\infty} T(\mathcal{E})\, \overbrace{D_1(\mathcal{E}+eV)}^{\approx D_1(\mathcal{E})} D_2(\mathcal{E}) \times$$

$$\times \underbrace{\left[f(\mathcal{E}+eV)\cdot(1-f(\mathcal{E})) - f(\mathcal{E})(1-f(\mathcal{E}+eV))\right]}_{f(\mathcal{E}+eV)-f(\mathcal{E})} d\mathcal{E}$$

$$f(\mathcal{E}+eV) - f(\mathcal{E}) = \frac{1}{e^{\frac{\mathcal{E}+eV}{kT}}+1} - \frac{1}{e^{\frac{\mathcal{E}}{kT}}+1}$$

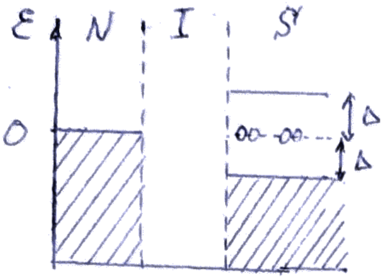
$$\approx \frac{1}{e^{\frac{\mathcal{E}+eV}{kT}}} - \frac{1}{e^{\frac{\mathcal{E}}{kT}}} = e^{-\frac{\mathcal{E}}{kT}}\left(e^{-\frac{eV}{kT}} - 1\right) \propto -eV$$

$I \propto V$ - Ohm's law



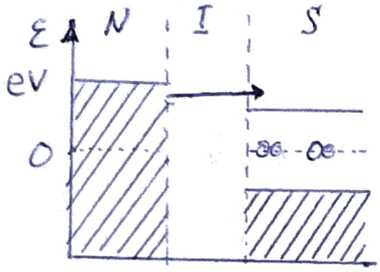
3.3 Contact between a normal metal and a SC (N-I-S)

$V=0$, $T=0$. Fermi levels will equilibrate



Pairs are heavy ⇒ tunneling probability is low.
For quasiparticles there are no states available within the gap ($D_S(\mathcal{E})=0$, for $|\mathcal{E}|<\Delta$).
Therefore, no current (for $|V|<\frac{\Delta}{e}$).

$V>\frac{\Delta}{e}=V_g$ (gap voltage)



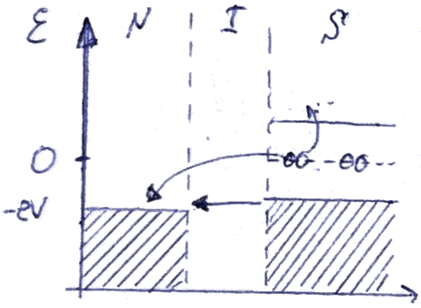
Similarly to N-I-N junction,

$$I\propto\int_{-\infty}^{+\infty} T(\mathcal{E})\cdot \underbrace{D_N(\mathcal{E}+eV)}_{\approx D_N(0)}\cdot D_S(\mathcal{E})\cdot\left[f(\mathcal{E}+eV)-f(\mathcal{E})\right]\cdot d\mathcal{E}$$

current of quasiparticles

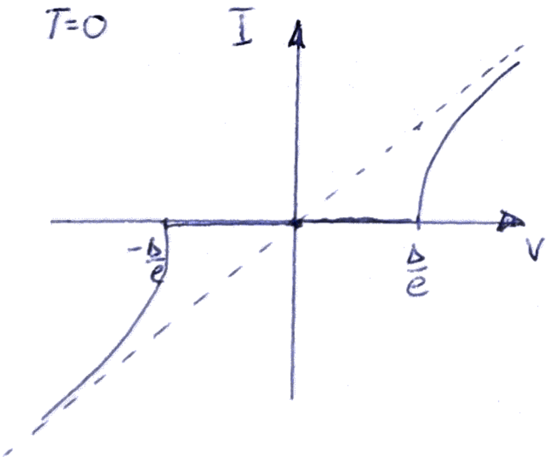
For $V\gg V_g$, $I\propto V$

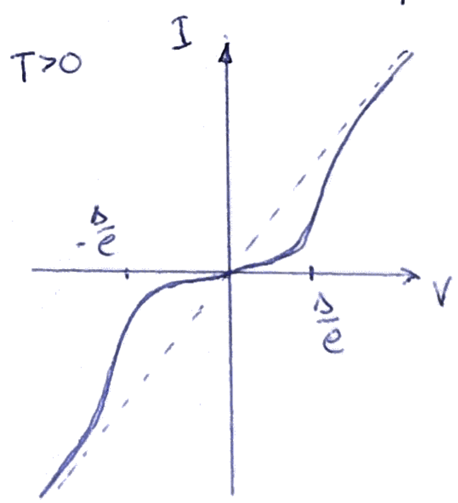
$V<-\frac{\Delta}{e}=-V_g$



Pairs can break: one electron gains energy Δ, another electron tunnels and gives energy $|eV|$.

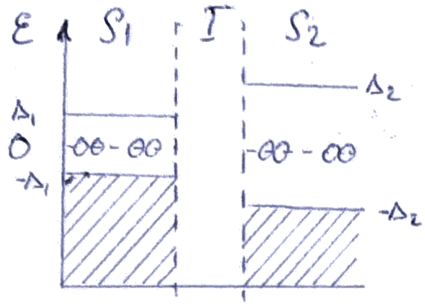
If $T>0$, there some quasiparticles that can tunnel even for $|V|<\left|\frac{\Delta}{e}\right|$.



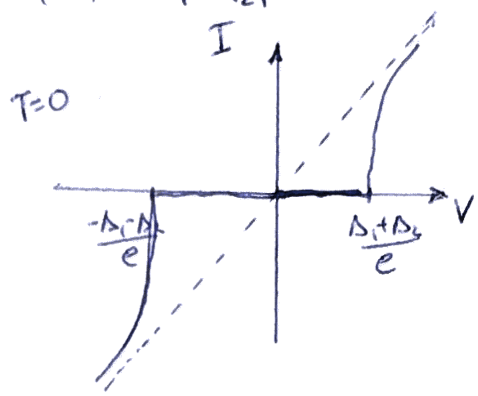


3.4 Contact between two superconductors (S-I-S)

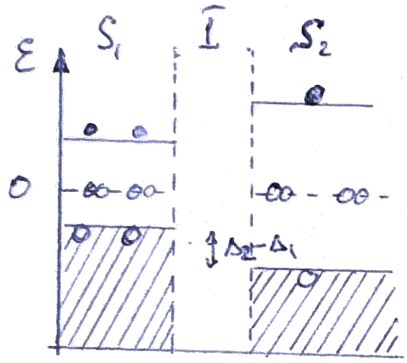
$T=0$, $V=0$



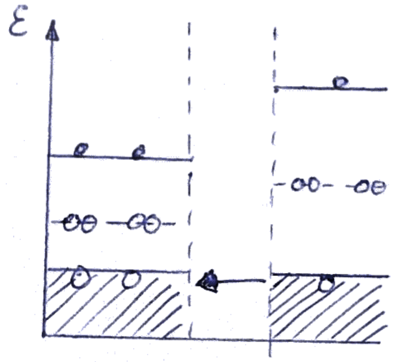
Current of quasiparticles is possible if $|eV| > |\Delta_1 + \Delta_2|$



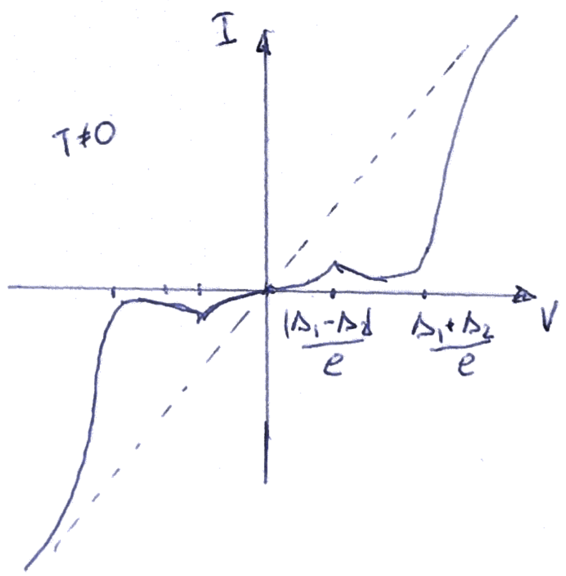
$T>0$, $V=0$



$T>0$, $eV = \Delta_2 - \Delta_1$

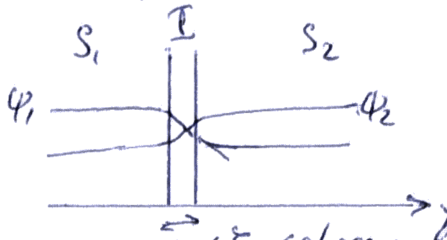


For $T>0$, there will be a spike of current for $|eV| = |\Delta_2 - \Delta_1|$ because of high density of state $D_S(\varepsilon)$ in a superconductor for $\varepsilon = \Delta$.



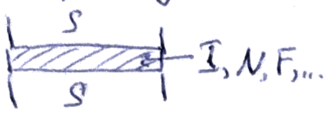
3.5 Josephson junction (1962)

If the barrier between two SCs is thin, cooper pairs can tunnel. The wavefunctions of pairs are intertwined.



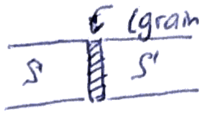
Types of Josephson junctions (JJs)

barrier

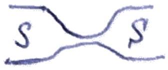


point contact

structural defect (grain boundary, ion irradiation)



constriction (Dayem bridge)



Superconducting current: $\vec{j}_s = \frac{(2e)\hbar}{2m}|\psi_0|^2 \nabla\Theta - \frac{2(2e)^2}{2m}|\psi_0|^2 \cdot \vec{A}$ $\quad (\psi = \psi_0 e^{i\Theta(\vec{r})})$

No external mag. field, no induced mag. field (j_s is small) $\Rightarrow \vec{A} = 0$

Barrier is thin, so there is no „gradient" of phase, only a „jump" $\delta = \Theta_2 - \Theta_1$

j_s is a function of δ: $j_s = f(\delta)$

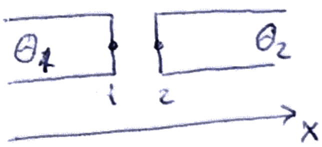
f is periodic, $T = 2\pi$

if $t \to -t$ (time inversion) $\Rightarrow -j = f(-\delta) \Rightarrow f(\delta) = -f(-\delta)$

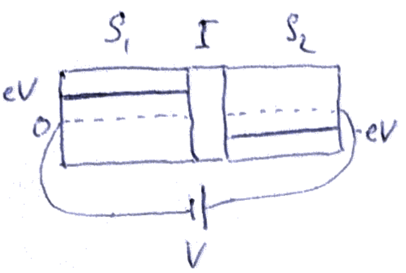
$\Rightarrow j = j_0 \cdot \sin\delta + j_1 \cdot \sin(2\delta) + j_2 \cdot \sin(3\delta) + \ldots$

For many cases (e.g. bJJ), $j = j_0 \cdot \sin\delta$ is a good approximation ($j_0 \sim 1\,mA$)

External magnetic field can be also included: $\delta = \Theta_2 - \Theta_1 - \frac{2e}{\hbar}\int_1^2 A_x \cdot dx$



3.6 Josephson equations (derivation after Feynman)



When the voltage V is applied across the JJ, the difference between energy levels is $(2e)\cdot V$

pair charge

Reminder from quantum mechanics: $i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi$

$$\psi(t) = \sum_\alpha C_\alpha(t)\cdot \psi_\alpha, \quad \alpha\text{-stationary state}$$

$$\sum_\alpha \psi_\alpha \cdot i\hbar \frac{\partial C_\alpha(t)}{\partial t} = \sum_\alpha C_\alpha(t)\cdot \hat{H}\psi_\alpha$$

$$i\hbar \frac{\partial C_\beta(t)}{\partial t} = \sum_\alpha C_\alpha(t)\cdot \int \psi_\beta^* \hat{H}\psi_\alpha \cdot d\vec{r}$$

Here we have two-level system,

$$\begin{cases} i\hbar \frac{d}{dt} C_1(t) = eV\cdot C_1(t) + K\cdot C_2(t) \\ i\hbar \frac{d}{dt} C_2(t) = -eV\cdot C_2(t) + K\cdot C_1(t) \end{cases}$$

$|C_1|^2 \to$ probability to find a pair in the state „1"
$|C_2|^2 \to$ probability to find a pair in the state „2"
$|C_1|^2 = n_{s_1}, \quad |C_2|^2 = n_{s_2}$.

$$C_1 = \sqrt{n_1}\, e^{i\theta_1}, \quad C_2 = \sqrt{n_2}\, e^{i\theta_2}$$

$$\begin{cases} i\hbar \frac{1}{2\sqrt{n_1}}\cdot \dot{n}_1 e^{i\theta_1} - \hbar\dot{\theta}_1 \sqrt{n_1}\cdot e^{i\theta_1} = eV\sqrt{n_1}e^{i\theta_1} + K\sqrt{n_2}\cdot e^{i\theta_2} \quad \Big| \cdot \sqrt{n_1}\cdot e^{-i\theta_1} \\ i\hbar \frac{1}{2\sqrt{n_2}}\cdot \dot{n}_2 e^{i\theta_2} - \hbar\dot{\theta}_2\sqrt{n_2}\, e^{i\theta_2} = -eV\sqrt{n_2}e^{i\theta_2} + K\sqrt{n_1}e^{i\theta_1} \quad \Big| \cdot \sqrt{n_2}\cdot e^{-i\theta_2} \end{cases}$$

$$\delta = \theta_2 - \theta_1$$

$$\begin{cases} \frac{i\hbar}{2}\dot{n}_1 - \hbar\dot{\theta}_1 n_1 = eV\cdot n_1 + K\sqrt{n_1 n_2}\, e^{i\delta} \\ \frac{i\hbar}{2}\dot{n}_2 - \hbar\dot{\theta}_2 n_2 = -eV\cdot n_2 + K\sqrt{n_1 n_2}\, e^{-i\delta} \end{cases} \Rightarrow \text{Im:} \begin{cases} \frac{\hbar}{2}\dot{n}_1 = K\sqrt{n_1 n_2}\cdot \sin\delta \\ \frac{\hbar}{2}\dot{n}_2 = -K\sqrt{n_1 n_2}\cdot \sin\delta \end{cases}$$

1^{st} Joseph. eq.

Current $I \propto \dot{n}_1 \propto -\dot{n}_2 \propto \sin\delta \Rightarrow \boxed{I = I_0\cdot \sin\delta}$

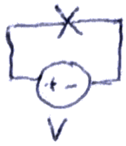
2^{nd} Joseph. eq

$$\text{Re:} \begin{cases} -\hbar\dot{\theta}_1 n_1 = eVn_1 + K\sqrt{n_1 n_2}\cdot\cos\delta \\ -\hbar\dot{\theta}_2 n_2 = eVn_2 + K\sqrt{n_1 n_2}\cdot\cos\delta \end{cases} \underset{n_1 \approx n_2}{\Longrightarrow} +\hbar\cdot\dot{\delta} = 2eV \Rightarrow \boxed{V = \frac{\hbar}{2e}\dot{\delta}} \Rightarrow \delta = \delta_0 + \underbrace{\frac{2eV}{\hbar}}\cdot t$$

$\omega \simeq 484$ MHz
$(V = 1\,\mu V)$

3.7 I-V characteristics of a Josephson junction

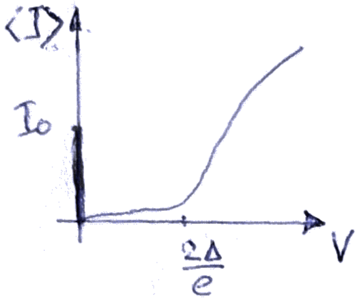
a) Apply constant voltage V



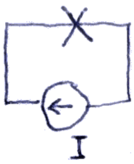
$V=0$, $I_S<I_0$: $I_S = I_0 \cdot \sin\delta$ ← constant supercurrent

$V>0$: $I_S = I_0 \cdot \sin\delta = I_0 \cdot \sin\left(\delta_0 + \frac{2eV}{\hbar}t\right)$ ← alternating supercurrent with $\langle I_S \rangle = 0$

$I_N = I_N(V)$ ← normal current of quasiparticles



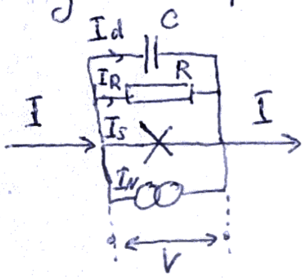
b) Apply constant current I



for $I \leq I_0$ → supercurrent

for $I > I_0$: $I = I_S + I_N$ ← sum of supercurrent and normal current

In this situation, one can use Resistively and Capacitively Shunted Junction model (RCSJ-model) of a JJ.
Stewart, McCumber, 1[illegible]



$I = I_S + I_R + I_d + I_N$

$I_S = I_0 \cdot \sin\delta$: supercurrent

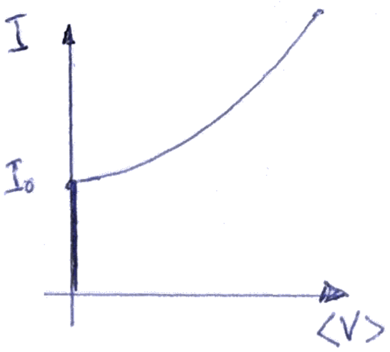
$I_d = C \cdot \dot{V}$: displacement current

$I_R = \frac{V}{R}$: dissipative current

$I_N(t)$: noise current at $T>0$

$V = \frac{\hbar}{2e} \cdot \dot{\delta}$: 2nd Josephson equation

Resulting I-V:



3.8 Shapiro spikes and Shapiro steps

a) Apply constant and alternating voltage

$$V = V_0 + a \cdot \cos(\omega t)$$

$$\dot{\delta} = \frac{2e}{\hbar} \cdot V \Rightarrow \delta = \delta_0 + \frac{2eV_0}{\hbar} \cdot t + \frac{2e \cdot a}{\hbar\omega} \cdot \sin(\omega t)$$

$$I_s = I_0 \cdot \sin\delta = I_0 \cdot \operatorname{Im}\left\{ e^{i\delta_0} \cdot e^{i\frac{2eV_0}{\hbar}t} \cdot e^{i\frac{2ea}{\hbar\omega}\cdot\sin(\omega t)} \right\}$$

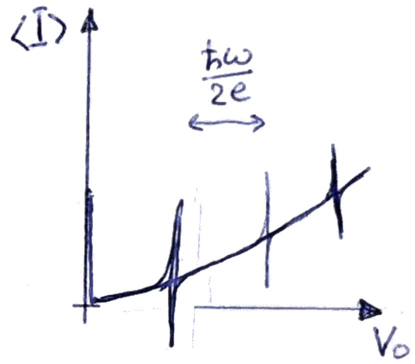
Using $e^{iC\sin x} = \sum_{n=-\infty}^{+\infty} J_n(C) \cdot e^{inx}$ ← Bessel function of the first kind, $J_{-n}(x) = (-1)^n J_n(x)$

$$I_s = I_0 \cdot \operatorname{Im}\left\{ e^{i\delta_0 + i\frac{2eV_0}{\hbar}t} \cdot \sum_{n=-\infty}^{+\infty} J_n\left(\frac{2ea}{\hbar\omega}\right) e^{in\cdot\omega t} \right\}$$

$$= I_0 \cdot \sum_{n=-\infty}^{+\infty} J_n\left(\frac{2ea}{\hbar\omega}\right) \cdot \sin\left[\delta_0 + \left(\frac{2eV_0}{\hbar} + n\omega\right)t\right]$$

$$= I_0 \cdot \sum_{n=-\infty}^{+\infty} (-1)^n \cdot J_n\left(\frac{2ea}{\hbar\omega}\right) \sin\left[\delta_0 + \left(\frac{2eV_0}{\hbar} - n\omega\right)t\right]$$

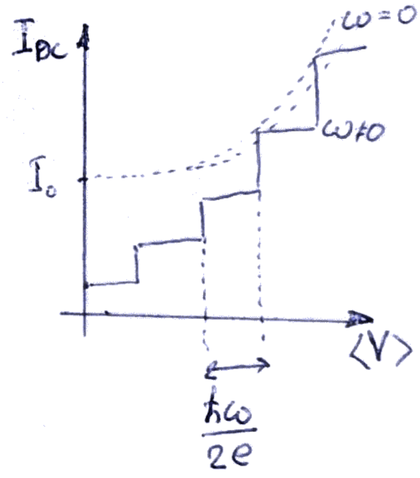
$\langle I_s \rangle = 0$, except when $V_0 = \frac{\hbar\omega}{2e} \cdot n$ ← Shapiro spikes



b) Apply constant and alternating current

$$I = I_{DC} + I_{ac} \cdot \cos(\omega t)$$

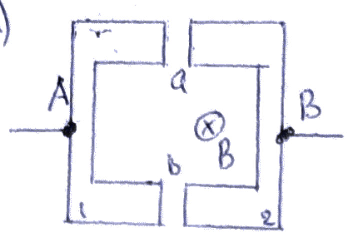
The resulting I-V curve will have Shapiro steps



If one knows ω precisely, one can measure voltage V precisely from the position of Shapiro steps.
This is the most accurate way to define voltage standard (Josephson normal)

3.9 Superconducting quantum interference devices (SQUIDs)

a)



$$I = I_a + I_b = I_0 \cdot \sin\delta_a + I_0 \cdot \sin\delta_b$$

inside a sc $j_s = 0$: $j_s = \frac{(2e)\hbar}{2m}|\psi_0|^2 \nabla\theta - \frac{2(2e)^2}{2m}|\psi_0|^2 \cdot \vec{A}$

$$\Rightarrow \nabla\theta = \frac{2e}{\hbar}\vec{A}$$

$$\theta_B = \theta_A + \frac{2e}{\hbar}\int_{AaB}\vec{A}d\vec{l} + \delta_a$$

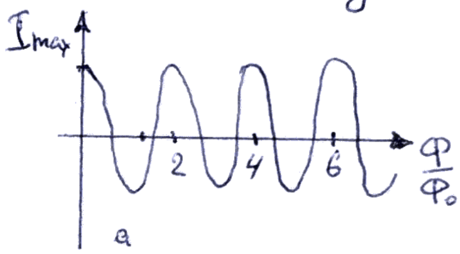
$$\theta_B = \theta_A + \frac{2e}{\hbar}\int_{AbB}\vec{A}d\vec{l} + \delta_B$$

$$\Rightarrow \delta_b - \delta_a = \frac{2e}{\hbar}\int_{AbB}\vec{A}d\vec{l} - \frac{2e}{\hbar}\int_{AaB}\vec{A}dl$$

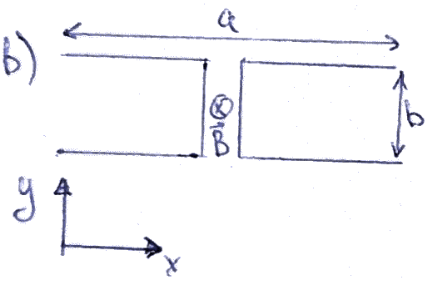
$$= \frac{2e}{\hbar}\oint\vec{A}d\vec{l} = \frac{2\pi}{\Phi_0}\cdot\Phi \quad \left(\Phi_0 = \frac{2\pi\hbar}{2e}\right)$$

$$I = I_0 \cdot 2\sin\frac{\delta_a+\delta_B}{2}\cdot\cos\frac{\delta_a-\delta_b}{2} \propto \cos\left(\pi\frac{\Phi}{\Phi_0}\right)$$

the maximum supercurrent is proportional to $\cos(\pi\frac{\Phi}{\Phi_0})$.
We built an interference device for precise measurements of magnetic flux



b)



$\vec{B} = \{0, 0, -B\}$
$\vec{B} = \nabla\times\vec{A} \Rightarrow \vec{A} = \{By, 0, 0\}$

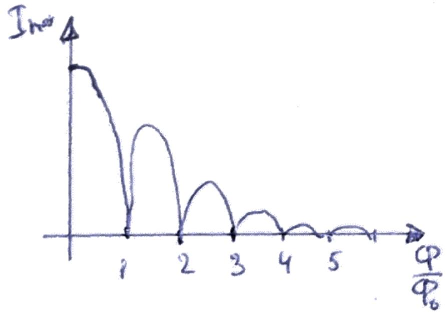
$$j_s = j_0 \cdot \sin\delta, \quad \delta = \underbrace{\theta_2 - \theta_1}_{\delta_0} - \frac{2e}{\hbar}\int_0^a A_x dx$$

$$j_s(y) = j_0 \cdot \sin\left(\delta_0 - \frac{2e}{\hbar}By\cdot a\right)$$

$$I_s = \int_0^c dz\cdot\int_0^b j_s(y)\cdot dy = j_0\cdot c\cdot\int_0^b \sin\left(\delta_0 - \frac{2e}{\hbar}Bya\right)\cdot dy$$

$$= -j_0\cdot c\cdot\left.\frac{\cos(\delta_0 - \frac{2e}{\hbar}Bya)}{-\frac{2e}{\hbar}Ba}\right|_0^b = \frac{j_0 c}{\frac{2e}{\hbar}Ba}\left[\cos\left(\delta_0 - \frac{2e}{\hbar}Bab\right) - \cos(\delta_0)\right]$$

$$= j_0\cdot\frac{c}{a}\cdot\frac{\hbar}{2e}\frac{1}{B}\cdot + 2\sin\left(\delta_0 - \underbrace{\frac{e}{\hbar}}_{\frac{\pi}{\Phi_0}}\underbrace{B\cdot ab}_{\Phi}\right)\cdot\sin\left(\underbrace{\frac{e}{\hbar}}_{\frac{\pi}{\Phi_0}}\underbrace{Bab}_{\Phi}\right)$$



$$= \underbrace{j_0\cdot c\cdot b}_{I_0}\cdot\frac{\Phi_0}{\pi\Phi}\cdot\sin\left(\frac{\pi\Phi}{\Phi_0}\right)\cdot\sin\left(\delta_0 - \frac{\pi\Phi}{\Phi_0}\right) = I_0\cdot\sin\left(\delta_0 - \frac{\pi\Phi}{\Phi_0}\right)\cdot\frac{\sin\frac{\pi\Phi}{\Phi_0}}{\frac{\pi\Phi}{\Phi_0}}$$

For given I_s, the phase δ_0 will adjust. The maximum current through the JJ is modulated by $\operatorname{sinc}\left(\frac{\pi\Phi}{\Phi_0}\right)$.