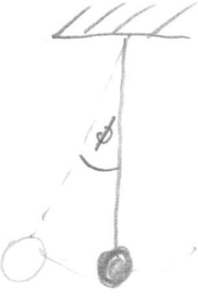
Oscillators in Physics

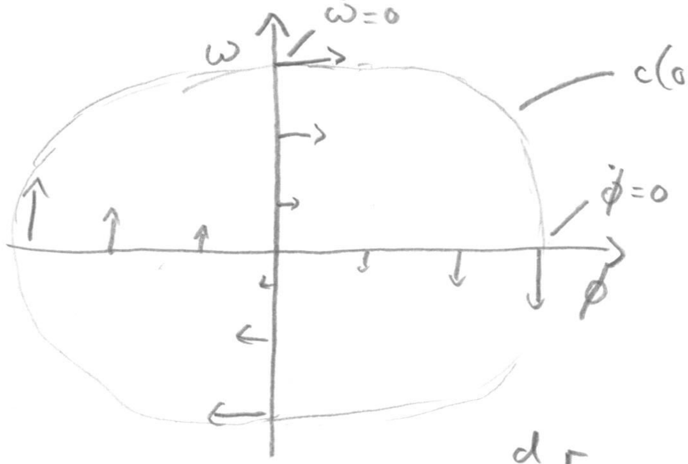


approx a harmonic oscillator

$$\dot\phi = \omega$$

$$\dot\omega = -\frac{g}{l}\sin(\varphi) \approx -\frac{g}{l}\phi$$

$$\Rightarrow \quad \phi(t) = A\sin\left(\sqrt{\frac{g}{l}}\,t + B\right)$$



closed orbits ≡ Energy conservation

$$E_{pot} = \frac{1}{2} mgl\phi^2$$

$$E_{kin} = \frac{1}{2} ml^2\omega^2$$

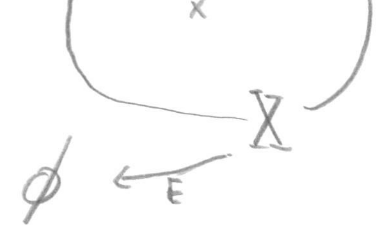
$$\frac{d}{dt}E_{tot} = mgl\phi\dot\phi + ml^2\omega\dot\omega = m(gl\phi\omega - gl\phi\omega) = 0$$

- Energy is converted to kinetic to potential & back
- Friction (dissipation) leads to damped oscillations

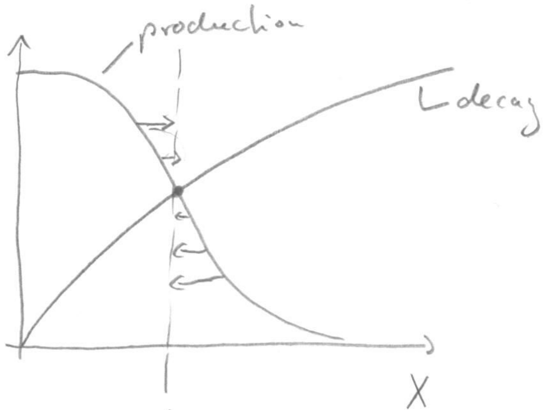
Oscillations in Biology

- production/degradation/modification of mRNA & proteins

delayed negative feed back



$$\frac{dX}{dt} = \underbrace{\frac{\alpha}{1+X^p}}_{\text{production}} - \underbrace{\frac{\beta X}{\gamma + X}}_{\text{decay}}$$

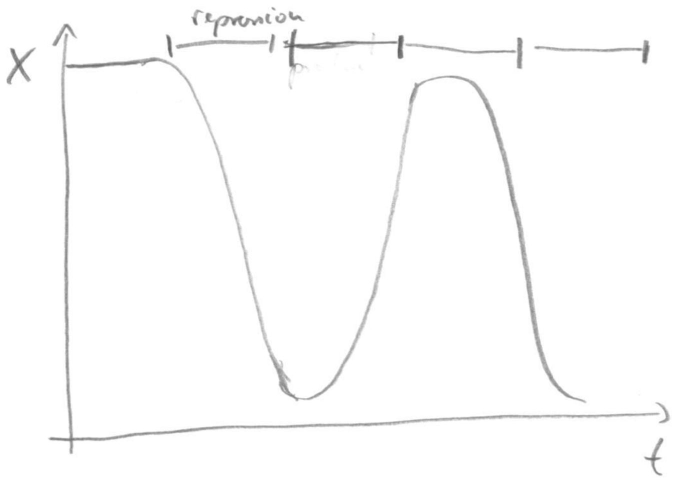


all flow is towards the mid line ⇒ converges to a stable fix point

⇒ no oscillations

$$\frac{dX}{dt} = \frac{\alpha}{1+X(t-\tau)^p} - \frac{\beta X(t)}{\gamma + X(t)}$$

↗ delay τ



⇒ computer illustration

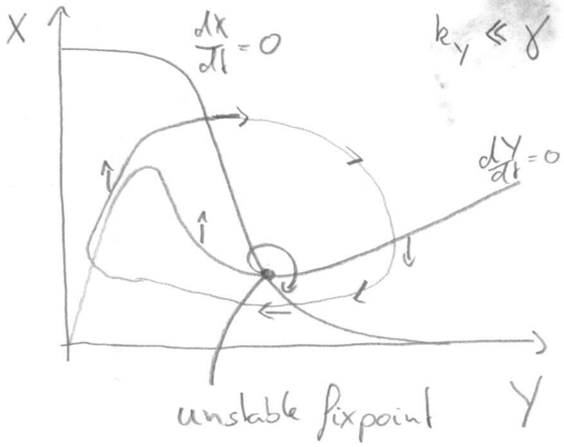
Basis of delay

- mRNA has to leave the nucleus
- be translated
- reenter the nucleus
- chain of intermediates (repressilator

No delay necessary for 2 components

$$\frac{dX}{dt} = \frac{\alpha}{1+Y^p} - k_x X \overset{!}{\equiv} 0 \Rightarrow X = \frac{\alpha}{k_x}\frac{1}{1+Y^p}$$

$$\frac{dY}{dt} = k_{yx} X - k_y Y - \gamma \frac{Y}{K + Y + K' Y^2} \overset{!}{\equiv} 0 \Rightarrow X = \frac{k_y}{k_{yx}} Y + \frac{\gamma}{k_{yx}} \frac{Y}{K+Y+K' Y^2}$$



$\Rightarrow$ self-sustained oscillations

Oscillator entrainment

$$\frac{dX}{dt} = \frac{\alpha}{1+Y^p} - k_x Y + A\sin(\omega t)$$

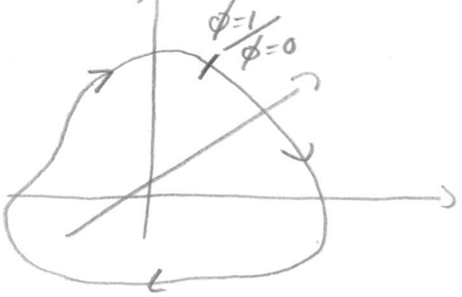
- if ω is close to the natural frequency ω_0, it will oscillate with ω
- stochastic oscillators get synchronized

Phenomenological oscillator model

- Volume & mass grow exponentially $\frac{dV}{dt} = \mu V$
- mother & daughter division times are different

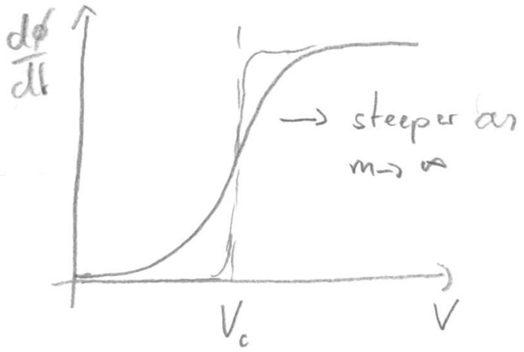
$$\underbrace{V_m + V_D}_{\text{volume @ division}} = V_m e^{\mu T_m} = V_D e^{\mu T_D} \Rightarrow e^{-\mu T_D} + e^{-\mu T_m} = 1$$

- size control: cell cycle is slowed down in cells that start small
- generic phase variable ϕ



$$\frac{d\phi}{dt} = \omega \frac{V^m}{V^m + V_c^m}$$

$$\frac{dV}{dt} = \mu V$$



Assumption: $\exists\, \phi_1$ when budding starts

- Volume increase of mother $\phi < \phi_1$
- " " daughter $\phi > \phi_1$

Volume changes in successive rounds of divisions

- $V > V_c \Rightarrow \frac{d\phi}{dt} \simeq \omega \qquad V_{n+1} = V_n \underbrace{e^{\frac{\mu\phi_1}{\omega}}}_{\text{volume increase prior to budding}}$

⇒ large cells grow further

⇒ division time is $T_\mu = \frac{1}{\omega}$

to determine the daughter volume

$$V_n e^{\frac{\mu\phi_1}{\omega}} + V^D_{n+1} = V_n e^{\mu/\omega}$$

$$\Rightarrow V^D_{n+1} = V_n \left(e^{\mu/\omega} - e^{\mu\phi_1/\omega}\right) < V_n \quad \text{if } e^{\mu/\omega} < 0$$

[mass doubling < division]

⇒ daughters shrink, until $V_n < V_c$

- $V_n < V_c$
  - growth until $V = V_c$ while $\phi = 0$

$$V_{\text{budding}} = V_c e^{\phi_1 \mu/\omega}$$

$$V_c e^{\phi_1 \mu/\omega} + V^D_{n+1} = V_n e^{\mu\left(\frac{1}{\mu}\log\frac{V_c}{V_n} + \frac{1}{\omega}\right)} = V_c e^{\mu/\omega}$$

$$V^D_{n+1} = V_c \left(e^{\mu/\omega} - e^{\phi_1 \mu/\omega}\right) = r V_c < V_c$$

⇒ Volume of successive daughters is constant