Magnetism in two dimensions and Mermin-Wagner theorem

by Frank Schreiber

Question: Does a lower dimension (e.g., 2D instead of 3D), i.e. "less neighbouring spins" change the ordering behaviour ?

Answer: Yes.

Fundamental statement

> "At any non-zero temperature, a one- or two-dimensional isotropic spin-S Heisenberg model with finite-range exchange interaction canc be neither ferromagnetic nor antiferromagnetic."
>
> see Mermin / Wagner, Phys. Rev. Lett. 17 (1966) 1133
> (note the assumptions "isotropic" and "finite-range"; these will be analysed later.)

Proof: The original proof is somewhat involved; we will not follow it here.

We will rather illustrate the physics behind the Mermin-Wagner theorem which is based on the idea that the excitation of spinwaves can destroy the magnetic order (here for a ferromagnet; argumentation would be different for antiferromagnet (with dispersion $E \sim k$))
Write the temperature-dependent magnetisation as

$$M(T) = M(T = 0) - \Delta M(T) \tag{1}$$

where $\Delta M(T)$ is the reduction of the magnetisation due to thermally excited spin-waves.

$\Delta M(T)$ is calculated according to the usual strategy, i.e. integrate over the density of states $N(E)$ of the excitations times their the probability for the thermal occupation (Bose-Einstein statistics)

$$\Delta M(T) \sim \int_0^\infty N(E)[1/(e^{E/k_BT} - 1)]dE \tag{2}$$

The important point to realise here is that $N(E)$ depends on the dimensionality of the system. Let us consider the general case of excitations with a dispersion

$$E \sim k^n \tag{3}$$

and a volume element in d-dimensional k space $\sim k^{d-1}dk$.

Using the above dispersion we write

$$k^{d-1} \sim E^{d-1/n} \tag{4}$$

Using

$$dk = (dk/dE)dE \tag{5}$$

we have

$$(dk/dE) \sim k^{-n}k \sim E^{-1}E^{1/n} \tag{6}$$

Thus, the volume element in k space ($k^{d-1}dk$) is expressed using the volume element in energy space

$$E^{(d-n)/n}dE \tag{7}$$

and the density of states is written as

$$N(E) \sim E^{(d-n)/n} \tag{8}$$

Therefore,
for $n = 2$ (dispersion of spin-waves in ferromagnetics (with Heisenberg-Hamiltonian))
and $d = 2$ (two dimensions)

$$N(E) = constant \tag{9}$$

and we have

$$\begin{aligned} \Delta M(T) &\sim \int_0^\infty const \;\; [1/(e^{E/k_BT} - 1)]dE \qquad (10) \\ &\sim T\int_0^\infty [1/(e^x - 1)]dx \qquad (11) \end{aligned}$$

Analyse this integral near the lower boundary (small x) using

$$e^x - 1 = x + ... \tag{12}$$

We find that

$$\int_0^\infty (1/x)dx \tag{13}$$

diverges logarithmically. This means that $\Delta M(T)$ diverges for finite T, which implies the breakdown of magnetic order, i.e. $M(T) = 0$ for $T > 0$.

The reason for the absence of magnetic order under the above assumptions is thus that at finite temperatures spin-waves are infinitely easy to excite, which destroys magnetic order.

Comments

1. assumption of ”isotropic interactions”

   If there is, e.g., an easy axis (anisotropy) in the system, the dispersion has a different form, such as, e.g.

$$E = A + Dk^2 \tag{14}$$

with $A = const$. This translates into an integral (using the same arguments regarding density of states etc as above) of the type

$$\Delta M(T) \sim \int_A^\infty const \;\; [1/(e^{E/k_B T} - 1)]dE \tag{15}$$

Since the lower boundary is now shifted ($A > 0$) the integral does not diverge at this boundary and the magnetic order is thus stabilised by anisotropy.

2. assumption of "short-range interactions"

   If we assume longer-range interactions (e.g., dipolar) and a different dispersion, e.g., such as

$$E \sim k^{1/2} \qquad \text{for small } k \tag{16}$$

   we find

$$N(E) \sim E^3 \tag{17}$$

   and the integral

$$\Delta M(T) \sim \int_A^\infty E^3 [1/(e^{E/k_B T} - 1)]dE \tag{18}$$

   does not diverge at its lower boundary.

   Thus, magnetic order may be stabilised in this model.

3. assumption of dimension $d = 2$ (or smaller)

   Consider the dimension d as a continuous parameter with $d = 2 + \epsilon$.

   Assuming for simplicity the conventional dispersion $E \sim k^2$ and using the above arguments this gives

$$N(E) \sim E^{\epsilon/2} \tag{19}$$

   which leave to an integral for $\Delta M(T)$ of the type

$$\Delta M(T) \;\; \sim \;\; \int_0^\infty E^{\epsilon/2} [1/(e^{E/k_B T} - 1)]dE \tag{20}$$

$$\sim \;\; T \int_0^\infty dx/(x^{1-(\epsilon/2)} + ...) \tag{21}$$

   which does not diverge.

   Again, a deviation from the original assumptions by Mermin and Wagner, i.e. in this case no strict two-dimensionality, but rather slightly higher (e.g. a film with a finite thickness, i.e. not only a monolayer) stabilises the order.

4. comparison with experiments on real-world samples

   While the assumptions (isotropic and short-range) is usually not strictly fulfilled, it is hard to confirm Mermin-Wagner in a real system.

   Nevertheless, the theorem provides an important benchmark and gives a qualitative explanation why the ordering temperature T_c is usually reduced for thinner films.

   see, e.g., Schneider et al., Phys. Rev. Lett. 64 (1990) 1059
   (T_c depends on thickness for ultrathin films)